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1987 J. Phys. A: Math. Gen. 20 3209

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## Scaling for period doubling sequences with correlated noise

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Received 2 June 1986, in final form 30 September 1986

**Abstract.** The scaling behaviour of the period doubling sequences, perturbed by correlated noises, is investigated using a renormalisation group. The renormalisation transformation is achieved by a path integral formalism which allows a necessary generalisation of the random perturbation notion. For a stationary and weakly correlated (in a sense specified in this paper) Gaussian random perturbation it is proved that the scaling behaviour depends only on the same universal constant  $\kappa = 6.619 \dots$  as for uncorrelated perturbations.

### 1. Introduction

The onset of turbulence after period doubling sequences (Feigenbaum's scenario) is now well known [1-5]. For one-parameter families of maps on an interval such as  $x \rightarrow 1 - \mu x^2$ ,  $x \in [0, 1]$ ,  $\mu \in [0, 2]$ , we observe a cycle of period  $2^i$ , stable for  $\mu$  in the interval  $[\mu_i, \mu_{i+1}]$ . For  $\mu = \mu_{i+1}$  this cycle becomes unstable through a subharmonic bifurcation and a stable cycle of period  $2^{i+1}$  appears (the limit of the sequence  $\mu_n$  ( $\lim_{n \rightarrow \infty} \mu_n = \mu_\infty$ ) exists). The lengths of the intervals  $[\mu_i, \mu_{i+1}]$  scale as  $|\mu_i - \mu_{i+1}| / |\mu_{i-1} - \mu_i| \approx \delta^{-1}$ , near  $\mu = \mu_\infty$ ;  $\delta = 4.669 \dots$  is the universal Feigenbaum constant. If this one-parameter family of maps is perturbed by an external noise, only a finite number of doubling bifurcations is observable. For a decorrelated random perturbation several authors [6, 7] bring out a new universal constant:  $\kappa = 6.619 \dots$  which can be interpreted in the following way: the attractor for  $\mu$  (near  $\mu_\infty$ ) perturbed by an uncorrelated noise of amplitude  $\varepsilon \ll 1$  and the attractor for  $\mu' = \mu_\infty + \delta(\mu - \mu_\infty)$  perturbed by an uncorrelated noise of amplitude  $\kappa\varepsilon$  are similar except for a scaling factor  $\lambda \approx -0.4$  ( $\lambda$  is the scaling factor of the period doubling cascade without noise). In particular, for  $\mu = \mu_\infty$  the attractor when the noise amplitude is  $\varepsilon$  has twice as many bands as when the noise amplitude is  $\kappa\varepsilon$ .

However, if the random perturbation is temporally correlated, the universal constant, if it exists, can differ from  $\kappa$ . It can even take several values. In particular, if the perturbation is not random (it can be seen as a very strongly correlated random perturbation) the universal constant, calculated by a renormalisation group technique, is  $\delta = 4.669 \dots$  and not  $\kappa$ . Moreover the study by Arneodo [8] of a quasiperiodic perturbation (which can be seen as a quasiperiodic correlated random perturbation) indicates that there is a continuum of universal constants.

In this paper, we investigate the problem of the Feigenbaum scenario perturbed by a correlated noise, using the renormalisation group technique. To define correctly the renormalisation transformation, it is no longer possible to consider the external random perturbations to be of the form  $\xi_{n+1}(x_n)$  with  $\{\xi_n, n \geq 1\}$  a process of random

maps. Fortunately the path integral formalism used by Shraiman *et al* [7] and Feigenbaum *et al* [9] allows us to generalise the notion of perturbation.

In § 2 we determine the renormalisation transformation for a large class of correlated Gaussian random perturbations; this transformation is a map  $\tau = (T, T_{n(f)})$  from a space  $F \times B$  in itself, with  $F$  the space of maps on the interval  $[-1, 1]$  and  $B$  a space associated with the correlation functions of the perturbation. For a stationary and weakly correlated Gaussian process which will be exactly defined in § 3, we prove the uniqueness of the universal constant  $\kappa = 6.619 \dots$ . So the scaling law behaviour is identical for an uncorrelated or for a weakly correlated random perturbation.

**2. The renormalisation transformation**

*2.1. Preliminaries*

First let us consider the simple case where the randomly perturbed one-parameter dynamical system is defined by the iterative equation:

$$X_{n+1} = F_{\mu, n+1}(X_n) \tag{1}$$

with  $\{F_{\mu, n}, n \geq 1\}$  a process of random maps defined by

$$F_{\mu, n}(X) = f_{\mu}(X) + \varepsilon \xi_n(X) \tag{2}$$

with  $\mu \rightarrow f_{\mu}$  a one-parameter family of maps on the interval (setting out a period doubling sequence),  $\{\xi_n, n \geq 1\}$  a process of Gaussian random maps on  $\mathbb{R}$  and  $\varepsilon$  a fixed parameter ( $\varepsilon \ll 1$ ) which determines the noise amplitude. Without loss of generality, we suppose that  $E(\xi_n(\cdot)) = O(E(\cdot))$  is the expectation value.

The renormalisation transformation  $T$  can be achieved, as in the deterministic problem ( $\varepsilon = 0$ ), by iterating the map  $F_{\mu, n}$  twice with a change of scale ( $\lambda \approx -0.4$ ).  $T$  is a map on the space of the random processes (2) and is defined by (the subscripts  $\mu$  are dropped)

$$(TF)_n = \lambda^{-1} \circ F_{2n+1} \circ F_{2n} \circ \lambda.$$

For an uncorrelated process ( $E(\xi_n(X) \cdot \xi_p(Y)) = 0$  if  $n \neq p$ ) and for sufficiently small  $\varepsilon$ , the renormalised process of random maps  $\{(TF)_n, n \geq 1\}$  is Gaussian if the terms of order  $\varepsilon^2$  are neglected; we have (to within  $O(\varepsilon^2)$ ):

$$(TF)_n = \lambda^{-1} \circ f \circ f \circ \lambda + \varepsilon \lambda^{-1} \circ [\xi_{2n+1} \circ f \circ \lambda + (f' \circ f \circ \lambda)(\xi_{2n} \circ \lambda)]. \tag{3}$$

Thus, to within  $O(\varepsilon^2)$ , this process has the same statistical properties as the Gaussian process  $\{G_n, n \geq 1\}$  defined by

$$G_n = \lambda^{-1} \circ f \circ f \circ \lambda + \varepsilon \lambda^{-1} \circ \zeta_n$$

where  $\{\zeta_n, n \geq 1\}$  is an uncorrelated Gaussian process of random maps such as

$$E(\zeta_n(X)) = 0$$

$$E(\zeta_n^2(X)) = E([\xi_{2n+1} \circ f \circ \lambda(X) + (f' \circ f \circ \lambda)(\xi_{2n} \circ \lambda)(X)]^2)$$

for all  $X$  belonging to the interval.

More generally, for any random process  $\{\xi_n, n \geq 1\}$  we look for a random process  $\{\zeta_n, n \geq 1\}$  with the same distribution law as the process  $\{\xi_{2n+1} \circ f \circ \lambda + (f' \circ f \circ \lambda) \times (\xi_{2n} \circ \lambda), n \geq 1\}$  (i.e. the moments of these two processes are identical).

There are two kinds of scaling laws for the dynamical system (1). One is the scaling on the family  $\mu \rightarrow f_\mu$  (the deterministic part) and the other one is the scaling of the moments of the random perturbation. From (3) we can deduce the renormalisation transformation  $\tau = (T, T_{n(f)})$ :  $T$  acts on the deterministic part  $f$  ( $Tf = \lambda^{-1} \circ f \circ f \circ \lambda$ ) and  $T_{n(f)}$  (which depends on the map  $f$  on which  $T$  acts) acts on the moments of the perturbation. For the uncorrelated equidistributed Gaussian case we have (to within  $O(\varepsilon^3)$ ):

$$(T_{n(f)}R)(x) = \lambda^{-2} \{R(f(\lambda x)) + [f'(f(\lambda x))]^2 R(\lambda x)\}$$

with  $R(x) = \varepsilon^2 E(\xi_i(x), \xi_i(x))$  for all  $i \in \mathbb{N}^*$ .

### 2.2. Generalisation of the random perturbation

It is conceptually more satisfactory (and necessary for the correlated noises as we shall see) to construct the renormalisation transformation by using a path integral statistical method [7, 9-11]; a great advantage of this approach is to allow an easy generalisation of the random perturbation.

To use this technique, for the dynamical system characterised by the deterministic map  $f$ , we need to know the conditional probability density (or transition probability density):  $P(X_n = x_n | x_i \in [1, n-1], x_0, f)$ . This probability density is induced by the distribution law of the random process  $\{Y_n, n \geq 1\}$ , where by definition  $Y_n \equiv X_n - f(X_{n-1})$ . If the dynamical system is defined by an explicit equation like (1), the distribution law of  $\{Y_n, n \geq 1\}$  is determined by the law of the process of the random maps  $\xi_n$ , but it can also be given directly without specifying explicitly any process of this kind. Therefore it is not necessary to give an equation like (1), but to know the distribution law of the  $Y_n$  is sufficient.

We assume in the following sections that  $\{Y_n, n \geq 1\}$  is a Gaussian random process with  $E(Y_n) = 0$  for all  $n \geq 1$ ; in this case we only need the data of the cumulants  $R_{ij} \equiv E(Y_i Y_j)$  which can be functions of the  $X_n$ . From causality,  $P(X_n = x_n | x_i \in [1, n-1], x_0, f)$  depends only on the  $X_p$  with  $p \leq n$ . Subsequently we shall use a slightly more restrictive assumption: the covariance matrix coefficients  $R_{ij}$  will depend only on  $X_{i-1}$  and  $X_{j-1}$ ,  $R_{ij} \equiv R_{ij}(X_{i-1}, X_{j-1})$ , with the necessary condition:

$$R_{ij}(X_{i-1}, X_{j-1}) = R_{ji}(X_{j-1}, X_{i-1}). \tag{4}$$

The transition probability density is

$$\begin{aligned} P(X_n = x_n | x_i \in [1, n-1], x_0, f) &= (2\pi)^{-n} \int \prod_{j=1, n} dk_j \exp\left(-i \sum_{j=1, n} (x_j - f(x_{j-1}))\right) \\ &\times \exp\left(-\frac{1}{2} \sum_{l, m=1, n} R_{l, m}(x_{l-1}, x_{m-1}) k_l k_m\right). \end{aligned} \tag{5}$$

Conversely, if the covariance matrix  $R = [R_{ij}(X_{i-1}, X_{j-1})]$  of the Gaussian process  $\{Y_n, n \geq 1\}$  is positive definite for all values of the  $X_i$  it is possible to write the random perturbation explicitly in the same way as in (1): there is a lower triangular matrix  $A = [a_{ij}]$  such as  $R = A'A$  ( $'A$  is the transposed  $A$  matrix) and with  $a_{ij} \equiv a_{ij}(x_0, X_1, \dots, X_{j-1}, X_{i-1})$ . Therefore the dynamical system can be described by the

equations

$$\begin{aligned}
 X_1 &= f(x_0) + a_{11}(x_0)\lambda_1 \\
 X_2 &= f(X_1) + a_{21}(x_0; X_1)\lambda_1 + a_{22}(x_0; X_1)\lambda_2 \\
 &\dots \\
 X_n &= f(X_{n-1}) + a_{n1}(x_0; X_{n-1})\lambda_1 + \dots + a_{nn}(x_0, X_1, \dots, X_{n-1})\lambda_n \\
 &\dots
 \end{aligned}
 \tag{6}$$

where  $\{\lambda_n, n \geq 1\}$  is an uncorrelated equidistributed Gaussian process with  $E(\lambda_n) = 0$  and  $E(\lambda_n \cdot \lambda_p) = \delta_{np}$ .

However, if the covariance matrix  $R$  is degenerate we can no longer describe the randomly perturbed dynamical system by explicit equations such as (6). However, the dynamics of the system is always well defined by the transition probability density.

### 2.3. The renormalisation transformation

In what follows the assumptions about the noise are those of the previous subsection; in addition, the covariance matrix  $R$  is supposed to be of order  $\epsilon^2$  with  $\epsilon \ll 1$  (we are only dealing with small random perturbations). The renormalisation transformation  $\tau = (T, T_{n(f)})$  is obtained by decimation [7, 9]. Let us define the characteristic function  $P(k|x_0, f)$  of the probability density  $P(X_n = x_n|x_0, f)$ :

$$P_n(k|x_0, f) = \int dx_n P(X_n = x_n|x_0, f) \exp(-ikx_n)
 \tag{7}$$

with

$$P(X_n = x_n|x_0, f) = \int \prod_{i=1, n-1} dx_i P(X_n = x_n|x_i \in [1, n-1], x_0, f).$$

We integrate first on the  $k_{2i-1}$  and then on the  $x_{2i-1}$  to eliminate them in the formal expression of  $P_n(k|x_0, f)$  obtained from (6) and (7). After a rescaling with the factor  $\lambda \sim -0.4$  (in order to achieve the deterministic doubling transformation  $Tf = \lambda^{-1} \circ f \circ f \circ \lambda$ ) and using a saddle-node approximation ( $|R_{ij}| \sim \epsilon^2 \ll 1$ ) we achieved the renormalised characteristic function  $P_{n/2}(\lambda^{-1}k|x'_0, Tf)$ :

$$P_{n/2}(\lambda^{-1}k|x'_0, Tf) = P_n(k|x_0, f)
 \tag{8}$$

the formal expression of which is written in terms of the rescaled variables  $x'_i = \lambda^{-1}x_{2i}$  and  $k'_i = \lambda k_{2i}$ , the renormalised function  $Tf$  and a renormalised matrix  $T_{n(f)}R$ . The relation (8) points out the self-similarity of the dynamical system.

The renormalised covariance matrix  $T_{n(f)}R$  (which depends on the map  $f$ ) is, to within  $O(\epsilon^3)$ :

$$\begin{aligned}
 (T_{n(f)}R)_{ij}(x_{i-1}, x_{j-1}) &= \lambda^{-2} \{ R_{2i,2j}(f(\lambda x_{i-1}), f(\lambda x_{j-1})) + \dots \\
 &\quad + R_{2i-1,2j-1}(\lambda x_{i-1}, \lambda x_{j-1}) f'(f(\lambda x_{i-1})) f'(f(\lambda x_{j-1})) + \dots \\
 &\quad + R_{2i,2j-1}(f(\lambda x_{i-1}), \lambda x_{j-1}) f'(f(\lambda x_{j-1})) + \dots \\
 &\quad + R_{2i-1,2j}(\lambda x_{i-1}, f(\lambda x_{j-1})) f'(f(\lambda x_{i-1})) \}
 \end{aligned}
 \tag{9}$$

(where we write  $x_i$  instead of  $x'_i$ ).

It is important to note that the exact transformation is not known. However, if  $f = \Phi$ , with  $\Phi$  the fixed point of  $T$ , the expression (9) is in fact the exact tangent map

of the exact transformation at the fixed point  $(\Phi, 0)$  of the transformation  $T = (T, T_{n(f)})$ , where  $0$  is the null covariance matrix (i.e. there is no perturbation). The knowledge of this tangent map is sufficient to know the scaling laws due to a small random perturbation.

It can be checked that the space of covariance matrices with the property (4) is stable by  $T_{n(f)}$ :

if  $R$  is a positive matrix, so is  $T_{n(f)}R$ ;

if  $R_{ij}(x_{i-1}, x_{j-1}) = R_{ji}(x_{j-1}, x_{i-1})$  so  $(T_{n(f)}R)_{ij}(x_{ij}(x_{i-1}, x_{j-1})) = (T_{n(f)}R)_{ji}(x_{j-1}, x_{i-1})$ .

At this point, let us notice that if the dynamical system is defined by an equation like (1), it cannot be so for the renormalised dynamical system.

If we assumed the random process  $\{Y_n, n \geq 1\}$  to be a stationary process:

$$\begin{aligned} R_{i+n,i}(x, y) &\equiv R_n(x, y) && \text{for } n \neq 0 \\ R_{i,i}(x, y) &\equiv R_0(x) \\ R_{-n}(x, y) &= R_n(y, x) \end{aligned} \tag{4'}$$

the expression of the renormalisation transformation  $T_{n(f)}$  is deduced from (9). We obtain

$$\begin{aligned} (T_{n(f)}R)_0(x) &= \lambda^{-2}\{R_0(f(\lambda x)) + R_0(\lambda x)[f'(f(\lambda x))]^2 + 2R_1(f(\lambda x), \lambda x)f'(f(\lambda x))\} \\ (T_{n(f)}R)_n(x, y) &= \lambda^{-2}\{R_{2n}(f(\lambda x), f(\lambda y)) + R_{2n}(\lambda x, \lambda y)f'(f(\lambda x))f'(f(\lambda y)) + \dots \\ &\quad + R_{2n+1}(f(\lambda x), \lambda y)f'(f(\lambda y)) + R_{2n-1}(\lambda x, f(\lambda y))f'(f(\lambda x))\} \\ &\text{for } n \neq 0. \end{aligned} \tag{10}$$

Now we need to define more precisely the space on which  $\tau = (T, T_{n(f)})$  acts. Let  $F$  be the space of the maps  $f$  on the interval  $[-1, 1]$  and  $B(\Omega)$  the space of maps from  $\Omega$ , an open neighbourhood in  $\mathbb{C}^2$  of the set  $[-1, 1] \times [-1, 1]$ , into  $1^\infty = \{\{x_n\}_{n \in \mathbb{Z}} \mid x_n \in \mathbb{C}, \sup |x_n| < \infty\}$  the space of complex numbers bounded sequences, so that

$$\begin{aligned} b_n(x, y) &= b_{-n}(y, x) && \text{for } n \neq 0 && b \in B(\Omega) \\ b_0(x, y) &= b_0(x) && && (x, y) \in \Omega. \end{aligned}$$

The transformation  $\tau = (T, T_{n(f)})$  is a map on the space  $F \times B(\Omega)$ .  $T$  has the trivial fixed point  $(\Phi, 0)$  where  $0$  is the null map of  $B(\Omega)$  and  $\Phi$  the known fixed point of  $T$ . The tangent map of  $\tau$  at  $(\Phi, 0)$  is  $D\tau_{(\Phi,0)} = (DT_{(\Phi)}, DT_{n(\Phi)(\Phi,0)} \equiv T_{n(\Phi)})$ . In the third section we will consider some restriction of  $\tau$  on subspaces  $F \times B'(\Omega)$  of  $F \times B(\Omega)$  with  $B'(\Omega)$  some subspaces of  $B(\Omega)$ .

### 3. Spectrum of the tangent map $DT_{n(\Phi)} = T_{n(\Phi)}$

The critical exponents due to the Gaussian stationary random perturbation, characterised by the covariance functions  $R_n(x, y)$  (4'), are determined by the eigenvalues of  $DT_{n(\Phi)} = T_{n(\Phi)}$  (defined by (10)) of modulus larger than or equal to 1. This spectrum depends on the space on which  $T_{n(\Phi)}$  acts; in other words, the critical exponents may not be the same if the perturbation is a weakly or a strongly correlated noise.

In what follows we investigate a category of weakly correlated random perturbations. For this purpose we introduce some notation.

Let  $H_\theta$  be the subspace of  $l^\infty(\mathbb{Z})$  defined by

$$H_\theta = \{h = \{h_n\}_{n \in \mathbb{Z}}; h_n \in \mathbb{C}, |h_n| < \infty \forall n \in \mathbb{Z} \text{ and } \lim_{|n| \rightarrow \infty} |h_n| \exp(\theta|n|) = 0\}$$

with  $\theta$  some positive real number. Equipped with the norm

$$\|h\|_\theta = \sup_{n \in \mathbb{Z}} \{|h_n| \exp(\theta|n|)\}$$

$H_\theta$  is a Banach space.

Let  $B(\Omega) \subset \mathbf{B}(\Omega)$  be the space of analytical functions from the open set  $\Omega$  into  $H_\theta$ :

$$B(\Omega) = \{b: \Omega \subset \mathbb{C}^2 \rightarrow H_\theta, b_n(x, y) = b_{-n}(x, y) \text{ and } b_0(x, y) = b_0(x)\}.$$

Equipped with the norm  $\|b\| = \sup_{\Omega} \|b(x, y)\|_\theta$ ,  $B(\Omega)$  is a Banach space. We will consider the space  $B_\Delta \equiv B(\Omega(\Delta))$  where

$$D(\Delta) = \{z \in \mathbb{C}, |z - y_0| < \Delta \text{ for some } y_0 \in [-1, 1] \times [-1, 1]\}.$$

Let us note that if the stationary Gaussian random perturbation does not depend on the values  $x_i$ , then we can define the spectral density  $\rho(z) = \sum_{n \in \mathbb{Z}} R_n z^n$  where  $z = \exp(i2\pi k)$ . The choice of the space  $B_\Delta$  imposes the analyticity of  $\rho$  on a ring centred on the circle  $|z| = 1$ , its width being smaller than  $\theta$ .

*Theorem 1.* For sufficiently small  $\Delta$ , the restriction of  $T_{n(\Phi)}$  to  $B_\Delta$ , denoted by  $T_{n(\Phi)}$ , is a compact operator from  $B_\Delta$  into itself.

*Proof.* Let  $T_q = T_{n(\Phi)} \circ P_q$  where  $P_q$  is a projector on  $B_\Delta$ :

$$(P_q b)_n = \begin{cases} b_n & \text{if } |n| \leq q \\ 0 & \text{if } |n| > q \end{cases} \quad b \in B_\Delta.$$

- (1) For  $\Delta$  sufficiently small the action of  $T_q$  enlarges the analyticity domain (see [5]).
- (2) If  $V$  is a bounded set of  $B_\Delta$ , so is  $T_q V$  for the compact convergence topology.
- (3) The compactness of  $T_q$  is a consequence of Montel's theorem.
- (4)  $T_{n(\Phi)}$  is the limit of the compact operators  $T_q$  ( $T_{n(\Phi)} = \lim_{q \rightarrow \infty} T_q$  for the topology defined by the norm  $\|T_q\| = \sup_{\|b\| \leq 1} \|T_q b\|$ ), therefore  $T_{n(\Phi)}$  is a compact operator.

Therefore the spectral values of  $T_{n(\Phi)}$  of modulus larger than or equal to  $\sigma$ ,  $\sigma > 0$ , are eigenvalues and in finite number; the eigenspaces are finite dimensional. These eigenvectors and eigenvalues will be given by theorem 2; but first the following proposition is needed to give an important characteristic of the eigenvectors.

*Proposition.* Let  $R \in B_\Delta$  be an eigenvector of  $T_{n(\Phi)}$  with the eigenvalue  $K$ ,  $|K| \geq \sigma$ , with  $0 < \sigma < 1$ , then  $R_n = 0$  if  $n \neq -1, 0, 1$ .

*Proof.* If  $R \in B_\Delta$  then  $\forall (x, y) \in D(\Delta)$ ,  $\lim_{|n| \rightarrow \infty} |R_n(x, y)| \exp(\theta|n|) = 0$ . So  $\forall \nu > 0$  and  $\forall (x, y) \in D(\Delta) \exists N \in \mathbb{N}$  such that for all  $n$ ,  $|n| > N$ ,  $|R_n(x, y)| < \nu \exp(-\theta|n|)$ .

On the other hand, if  $R$  is an eigenvector of  $T_{n(\Phi)}$ , with the eigenvalue  $K$ :

$$\begin{aligned} |K| |R_n(x, y)| &= |(T_{n(\Phi)} R)_n(x, y)| \\ &= \lambda^{-2} |R_{2n}(\Phi(\lambda x), \Phi(\lambda y)) + R_{2n}(\lambda x, \lambda y) \Phi'(\Phi(\lambda x)) \Phi'(\Phi(\lambda y)) + \dots \\ &\quad + R_{2n+1}(\Phi(\lambda x), \lambda y) \Phi'(\Phi(\lambda y)) + R_{2n-1}(\lambda x, \Phi(\lambda y)) \Phi'(\Phi(\lambda x))|. \end{aligned}$$

Using the definition of the limit, for all  $n$  such that  $2|n| - 1 > N$  (i.e.  $|n| > (N + 1)/2$ ):

$$|K||R_n(x, y)| \leq \nu\{\alpha \exp(-\theta|2n|) + \beta[\exp(-\theta|2n + 1|) + \exp(-\theta|2n - 1|)]\}.$$

So  $|R_n(x, y)| \leq \nu a \exp(-2\theta|n|)$  with  $a \equiv [\alpha + \beta(\exp(-\theta) + \exp(\theta))]/\sigma$  and  $\alpha, \beta \in \mathbb{R}$  the upper bounds of the factors of  $R_{2n}(\cdot, \cdot)$ ,  $R_{2n+1}(\cdot, \cdot)$ ,  $R_{2n-1}(\cdot, \cdot)$ .

Iterating  $p$  times  $T_{n(\Phi)}$  on  $R$  we obtain

$$\forall (x, y) \in D(\Delta), \quad \forall \nu > 0 \quad \exists N$$

so that for  $|n| > [1 + (N - 1)/2p]$  then

$$|R_n(x, y)| \leq \nu a^p \exp(-2p\theta|n|).$$

In the limit  $p \rightarrow \infty$  we obtain

$$\forall (x, y) \in D(\Delta) \quad \text{and} \quad \forall n, |n| > 1 \quad \text{then} \quad R_n(x, y) = 0.$$

The eigenvalues of  $T_{n(\Phi)}$  of modulus larger than or equal to 1 and the corresponding eigenspaces are as follows.

*Theorem 2.* (a) The only eigenvalues of the operator  $T_{n(\Phi)}$  acting on  $B_\Delta$ , of modulus larger than or equal to 1, are  $\kappa^2 = 43.811 \dots$ ,  $\lambda^{-2}$ ,  $\lambda^{-1}$ , 1 and (b) the corresponding eigenspaces are, in the space  $B_\Delta$ , as follows.

(i) For the eigenvalue  $\kappa^2 = 43.811 \dots$  the subspace spanned by the vector  $R \in B_\Delta$  defined by  $R_n = 0$  for all  $n$  except  $R_0$  which is a solution of the eigenvalue equation corresponding to the uncorrelated problem:  $\kappa^2 R_0(x) = \lambda^{-2}[R_0(\Phi(\lambda x)) + R_0(\lambda x)[\Phi'(\Phi(\lambda x))]^2]$  ( $\kappa^2$  is the greatest eigenvalue of the uncorrelated problem [6, 7]).

(ii) For the eigenvalues  $\lambda^{t-2}$  ( $t = 0, 1, 2$ ) the finite-dimensional subspaces spanned for each  $t$  by the vectors  $R^{(t)} \in B$  defined by

$$\begin{aligned} R_n^{(t)} &\equiv 0 && \text{if } n \neq -1, 0, 1 \\ R_0^{(t)}(x) &= -a\Phi^t(x) + (a - 2A)\Phi^{t/2}(x)x^t \\ R_1^{(t)}(x, y) &= \sum_{p \in [-1, t]} A_p x^p \Phi^t(x) \Phi^{t-p}(y) \\ R_{-1}^{(t)}(x, y) &= R_1^{(t)}(y, x) \end{aligned}$$

with  $A_p \in \mathbb{C}$ ,  $A = \sum_{p \in [-1, t]} A_p$ ,  $a \in \mathbb{C}$ .

*Proof.* From the proposition we deduce that the only non-zero components of an eigenvector of  $T_{n(\Phi)}$ ,  $R \in B_\Delta$ , are  $R_{-1}$ ,  $R_0$ ,  $R_1$  which are solutions of the following eigenvalue equations:

$$KR_1(x, y) = \lambda^{-2}R_1(\lambda x, \Phi(\lambda y))\Phi'(\Phi(\lambda x)) \tag{11}$$

$$\begin{aligned} KR_0(x) &= \lambda^{-2}\{R_0(\Phi(\lambda x)) + R_0(\lambda x)[\Phi'(\Phi(\lambda x))]^2 + \dots \\ &\quad + 2R_1(\Phi(\lambda x), \lambda x)\Phi'(\Phi(\lambda x))\} \end{aligned} \tag{12}$$

with  $(x, y) \in D(\Delta)$ .

(a) Solutions of equation (11).

An obvious solution is  $R_1 \equiv 0$ .

Otherwise, if  $x \neq 0$  (so  $\Phi'(x) \neq 0$ ):

$$(11) \Leftrightarrow K\Phi'(\lambda x)R_1(x, y) = \lambda^{-2}R_1(\lambda x, \Phi(\lambda y))\Phi'(x)$$

because  $\Phi'(x) = \Phi'(\Phi(\lambda x))\Phi'(x)$ .



We note that the function  $R_1(x, y)/\Phi'(x)$  is homogeneous in  $x$ ; therefore, since  $R_1(x, y)$  is an analytical function:

$$R_1(x, y) = u(y)x^p\Phi'(x)$$

with  $p \geq -1$ , and  $u$  some analytical function. We recall that  $\Phi(x) = \Psi(x^2)$  and  $\Psi$  is such that  $\Psi'(t) \neq 0$  for  $t \in [0, 1]$ . Inserting this result in (11),  $u(y)$  must verify

$$Ku(y) = \lambda^{p-2}u(\Phi(\lambda y)) \tag{13}$$

whose solutions are

$$u(y) = \Phi^q(y) \quad \forall q \geq 0 \text{ and } K = \lambda^{p+q-2}.$$

Therefore the solutions of the eigenvalue equation (11) are

$$R_1^{(t)}(x, y) = \sum_{p \in [-1, t]} A_p x^p \Phi'(x) \Phi^{t-p}(y) \tag{14}$$

with  $A_p \in \mathbb{C}$  and  $K = \lambda^{t-2}$  where  $t = p + q \geq -1$ .

(b) Solutions of the equation (12) compatible with (14).

If  $R_1 \equiv 0$  then  $R_0(x)$  is a solution of the eigenvalue equation:

$$KR_0(x) = \lambda^{-2}\{R_0(\Phi(\lambda x)) + R_0(\lambda x)[\Phi'(\Phi(\lambda x))]^2\}. \tag{15}$$

((15) is the eigenvalue equation of the uncorrelated problem.) The solutions of (15) are the known eigenvector corresponding to the eigenvalue  $K = \kappa^2 = 43.811 \dots$  and the eigenvectors  $R_0 = S^{(t)}$   $t = 0, 1, 2$  corresponding to the eigenvalues  $K = \lambda^{t-2}$ ;  $S^{(t)}$  is given by

$$S^{(t)}(x) = -\sigma^{(t)}(\Phi(x)) + (\Phi'(x))^2 \sigma^{(t)}(x)$$

where  $\sigma^{(t)}$  is an analytical function on  $D(\Delta)$  satisfying

$$\sigma^{(t)}(\lambda x) = \lambda^t \sigma^{(t)}(x) \quad t = 0, 1, 2$$

then

$$S^{(t)}(x) = -a\Phi^t(x) + a[\Phi'(x)]^2 x^t \quad a \in \mathbb{C}.$$

If  $R_1$  is given by (14) then the solutions  $R_0(t)$  of (12) satisfy

$$\lambda^t R_0^{(t)}(x) = \{R_0^{(t)}(\Phi(\lambda x)) + R_0^{(t)}(\lambda x)[\Phi'(\Phi(\lambda x))]^2 + \dots + 2A\Phi(\lambda x)\Phi'(\Phi(\lambda x))\} \tag{16}$$

with  $A = \sum_{p \in [-1, t]} A_p$ .

Let  $R_0^{(t)}$  and  $Q_0^{(t)}$  be two solutions of (16); then the difference  $(R_0^{(t)} - Q_0^{(t)})$  is a solution of (15) with  $K = \lambda^t$ . Therefore the solutions of equation (16) are

$$R_0^{(t)}(x) = S^{(t)}(x) - 2A(\Phi'(x))^2 x^t$$

where  $-2A[\Phi'(x)]^2 x^t$  is a particular solution of equation (16).

Note about the proof of theorem 2. It is necessary to know the eigenvalues of the problem without correlation; the largest eigenvalue,  $\kappa^2 = 43.811 \dots$ , is numerically computed (see [6,7]); it can also be numerically checked, by a polynomial approximation, that the only eigenvalues larger than 1 are  $\kappa^2$  and  $\lambda^{t-2}$  ( $t = 0, 1, 2$ ) which eigenvectors are the  $S^{(t)}$  given in the proof.

Only the covariance matrices are significant for our problem: we are only interested in the cone of the positive matrices of the matrix space; so it is possible to eliminate

suppose that  $R$  is a positive eigenvector corresponding to  $\lambda^{-1}$ ;  $T_{n(\Phi)}R$  must be positive (see § 2.3) but it must also be negative since  $T_{n(\Phi)}R = \lambda^{-1}R$ , so there is a contradiction. On the other hand, we cannot eliminate all the other eigenvectors corresponding to  $\lambda^{-2}$  and 1 with similar arguments.

#### 4. Conclusion

In this survey we have determined the renormalisation group corresponding to a period doubling sequence perturbed by a Gaussian temporally correlated noise; for this purpose we have extended the notion of random perturbation: the perturbation is given by the distribution law of the random variables  $Y_n \equiv X_n - f(X_{n-1})$  and not by explicit random maps  $\xi_n(x)$ . For stationary Gaussian random processes (a reasonable assumption) and for sufficiently weak correlations, specified by the space  $B_\Delta$ , the scaling behaviour depends generically on the eigenvalue of  $T_{n(\Phi)}$  of largest modulus,  $\kappa^2$ , i.e. on  $\kappa = 6.619\dots$ , since  $T_{n(\Phi)}$  has a discrete spectrum; it is the same as without correlation.

On the other hand, with less restrictive correlation assumptions, we can no longer have the same conclusion; if we consider the restriction of the tangent operator  $T_{n(\Phi)}$  on a larger or another subspace of  $B(\Omega)$  than  $B_\Delta$  (but stable by the action of  $T_{n(\Phi)}$ ); this new operator can fail to be compact; for instance, the spectrum can be a continuum and the spectral subspaces be infinite dimensional: then even if  $|\kappa^2| \geq |K|$ , where the  $K$  are spectral values of the restriction of  $T_{n(\Phi)}$ , the scaling behaviour is not dominated by  $\kappa$  for a large class of perturbations (for example, see quasiperiodic correlations [8]). Finally we have to notice that for very strong correlations ( $R_n(x, y) = R(x, y)$  does not depend on  $n$ ),  $R(x, y) = \Phi(x)\Phi(y)$  is an eigenvector of  $T_{n(\Phi)}$  with the eigenvalue  $\delta^2$  (where  $\Phi$  is the fixed point of the deterministic renormalisation transformation  $T$  and  $\delta = 4.669\dots$  is the Feigenbaum universal constant): we find the deterministic scaling law again.

#### Acknowledgment

I thank P Collet for helpful discussions.

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